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AN ENTROPY CONSISTENT PARTICLE METHOD FOR NAVIER-STOKES EQUATIONS

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Abstract. *A variant of particle method with additional means for modeling discontinuous solutions is described. A possibility for its application even to incompressible viscous 2D fluid is considered.*

1 INTRODUCTION

The presenting computational method was developed as an endeavour to construct a through algorithm for study hierarchical models (combining kinetic and continuous medium approaches), f.i. space shuttle reentry. The particle method was chosen as basic one because it has proved its power for kinetic calculations (stochastic — connected to Boltzmann equation [2, 3, 4] and deterministic — connected to Kolmogorov – Fokker – Plank equation [6]). It also occurs that the proposing variant of particle method is very efficient for solving Navier – Stokes equations: it spreads shock waves over one particle, builds itself an "adaptive grid" (particles positions), conjugates naturally with stochastic particle method (Monte – Carlo simulations), transfers to multi dimensional case with low increase of operations number, can be easily parallelized because of its design.

The computational methods for gas dynamics is still one of the most important part of applied mathematics. The particle method proposed fifty years ago [1] is now more and more under consideration. It has a lot of advantages against finite difference or finite element methods. But usually one its disadvantage is underlined — its low accuracy. The reason is in the large smoothing of discontinuous solutions by traditional Particle – in – Cell (PIC) and Smoothed Particles Hydrodynamics (SPH) methods [5]. It seems that our method overcomes this disadvantage. So we would like to present the algorithm and to discuss a substantiation of the method.

The mathematical foundation of the method [3] lies in the following: solving the system of ordinary differential equations

$$D_t z(t) = \frac{1}{2} f_t(z(t)), \quad z\mid_{t=0} = z_0.$$

for trajectories $z(t, z_0)$ of particles distributed according prescribed initial measure μ_0, we get, that the linear functional $\mu_t : \psi \longmapsto \langle \psi, \mu_t \rangle = \int \psi(z(t,\xi))\mu_0(d\xi)$ is the solution of the system of equations

$$D_t \langle \psi, \mu_t \rangle = \langle D\psi[\frac{1}{2} f_t], \mu_t \rangle, \quad \mu_{+0} = \mu_0,$$
$$\langle \psi, \mu(t) \rangle = \int \psi(z) f_t(z) dz,$$

which is a generalized version of quasi–linear transport equation with the restriction that its solution possesses a density.

We have two obstacles on the way to a proper computational method. First, we need a procedure to recover the density by its measure, which is the main step of different variants of particle method (PIC, SPH etc.), and we propose such a new procedure, in the heart of which lies the principles restricting the spreading of the jumps (shock waves) by one particle. Second,we have to take into account that the nonlinear generalized problem has no unique solution and to take efforts for regularization of the ill-posed problem based upon so called "entropy" condition which is similar to maximum principle for 1D transport equation [7, 8, 9]. Our method differs from the known ones by the above two points that makes it possible to simulate accurately discontinuous solutions.

We present very much in brief the main features of the algorithm [13, 14, 18]. Choosing particles like rectangles (in 1D case) we approximate the density function $f_t(x)$ by a stepwise one that leads to a requirement of contiguity of these rectangles. The second requirement is their symmetry against their positions. While moving, our particle – rectangles can overlap or run away of each other. It is a consequence of discretization which gives rise to artificial oscillations. Our method struggles with this phenomenon by reconstructing the forms (not the areas) of the rectangles (i.e. their widths and heights) in order to satisfy the above requirements, as well as "entropy" condition, that gives specific rules of the algorithm.

The method was developed and tested for quasi – linear transport equation, gas dynamic system, shallow water models, incompressible fluid flows in 1D and 2D cases [13, 14, 15, 18]. It promises its very high quality.

2 LINEAR EQUATION

The mathematical foundation of the method lies in the following results on the generalized Liouville equation obtained in [10, 12, 3] for Vlasov equation. We'll write down the main statements of this approach without prove.

Let function $t \longmapsto F(t,z)$ belongs to a space $C([0,T], C^{(2)}(\mathbf{R^n} \longmapsto \mathbf{R^n}))$ and

$$\mid F(t,z) \mid \leq A \mid z \mid + B. \quad 0 \leq t \leq T \tag{1}$$

Consider in $\mathbf{R^n}$ a system of ordinary differential equations

$$D_t z(t) = F(t, z(t)), \quad z \mid_{t=0} = z_0. \tag{2}$$

Denote by $z(t, z_0)$ a solution of system (2). Under the condition (1) it exists at $[0,T]$ and is unique and the mapping

$$\mathbf{R^n} \longmapsto \mathbf{R^n} : z_0 \longmapsto z(t, z_0) \tag{3}$$

is continuously differentiable.

Let μ_0 be a probability Radone measure at $\mathbf{R^n}$, $z(t, z_0)$ be a solution of (2). Consider at $C_b(\mathbf{R^n})$ a linear functional

$$\mu_t : \psi \longmapsto \langle \psi, \mu_t \rangle = \int \psi(z(t,\xi))\mu_0(d\xi) \tag{4}$$

Lemma 1. *If the measure μ_0 has a property: at some $\alpha > 0$*

$$\int \mid \xi \mid^a \mu_0(d\xi) < \infty,$$

then the relation (4) *at each* $t > 0$ *define a probability Radone measure at* $\mathbf{R^n}$*, and as a function of* t *the measure belongs to a space* $C([0,T],M)$ *and for all* $t \in [0,T]$ *satisfies the estimation*

$$\int \mid z \mid^a \mu_t(dz) < \infty.$$

The measure μ_t can be defined not with the help of solution $z(t,\xi)$, but as solution of a generalized equation which we'll now derive. Differentiating (4), more precisely, considering a limit as $\Delta t \to 0$ of the relation

$$\frac{\langle \psi, \mu_{t+\Delta t}\rangle - \langle \psi, \mu_t\rangle}{\Delta t} = \frac{\int \psi(z(t+\Delta t,\xi)) - \psi(z(t,\xi))}{\Delta t}\mu_0(d\xi)$$

and taking into account the representation

$$z(t+\Delta t,\xi) = z(t,\xi) + \int_t^{t+\Delta t} F(s, z(s,\xi))ds = z(t,\xi) + F(s^*, z(s^*,\xi))\Delta t,$$

as well as the formula (4), we get for any function $\psi \in C_b^{(1)}(\mathbf{R}^n)$

$$D_t\langle \psi, \mu_t\rangle = \langle D\psi[F], \mu_t\rangle, \quad \mu_{+0} = \mu_0, \tag{5}$$

— *the generalized Liouville equation*, where the notation

$$D\psi[F](y) = D_\varepsilon \psi(y+\varepsilon F) \mid_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{\psi(y+\varepsilon F) - \psi(y)}{\varepsilon}.$$

is introduced.

Lemma 2. *At the above restrictions at function* F *the equation* (5) *with respect to measure* μ_t *has unique solution.*

Assume that at each $t > 0$ the measure μ_t defined by the formula (4) has a continuously differentiable density $f(z,t)$. According to definition of measure density we have for any function $\psi \in C_0^\infty(\mathbf{R}^n)$

$$\langle \psi, \mu_t\rangle = \int \psi(z) f(z,t)dz.$$

We see that the density of measure μ_t defined by the formula (4) satisfies the equation

$$D_t f + \sum_{1\le i\le n} D_i(F_i f) = 0, \quad D_i = \partial/\partial z_i, \tag{6}$$

which is called *Liouville equation* for system (1).

The above consideration leads to the following algorithm of the particle method for Liouville equation, i.e. for linear transport equation. Let's take an arbitrary natural N, construct some quadrature formula for initial measure μ_0 that means that with the help of a certain algorithm we define such N nodes $\xi_i^{\mu_0}$ that for any integrable function ψ an approximate equality

$$\int \psi(\xi)\mu_0(d\xi) \approx \frac{1}{N}\sum_{i=1}^{N}\psi(\xi_i^{\mu_0})$$

is fulfilled.

Each initial value of $\xi_i^{\mu_0}$ generates a trajectory $z(t, \xi_i^{\mu_0})$ as a solution of equation (2) and an expression

$$\frac{1}{N}\sum_{i=1}^{N}\psi(z(t, \xi_i^{\mu_0}))$$

is a right hand side of a quadrature formula, namely:

$$\approx \int \psi(z(t,\xi))\mu_0(d\xi),$$

which, in its turn, according to formula (4), is

$$= \int \psi(\xi)\mu_t(d\xi).$$

So we get the following statement.

Theorem. *If $\xi_i^{\mu_0}, i = 1, \ldots, N$, is a set of nodes of quadrature formula having an accuracy of order $O(1/N^\alpha)$, where a number α is defined by the choice of specific quadrature formula, then the solution of the generalized equation (5) has the same accuracy $O(1/N^\alpha)$ and is represented by the formula*

$$\int \psi(\xi)\mu_t(d\xi) \approx \frac{1}{N}\sum_{i=1}^{N}\psi(z(t, \xi_i^{\mu_0})),$$

where $z(t, \xi_i^{\mu_0})$ are exact solutions of equation (2) with initial values $\xi_i^{\mu_0}$.

We used the measure approximation by quadrature formula with the equal weights $1/N$ in all nodes. Our consideration will not change if we take a quadrature formula with the different weights:

$$\int \psi(\xi)\mu(d\xi) \approx \sum_{i=1}^{N} c_i\psi(\xi_i).$$

The given algorithm of particle method allows to build an approximate weak solution, i.e. to find out in which nodes in the time moment the measures $1/N$ (or c_i) are localized. It is the reason to require the symmetry against corresponding nodes of the measures distribution.

To look for a strong (classical) solution, it's necessary to recover by the measure distribution its density. To do this the number of nodes N has to be increased. Nevertheless at any fixed N, a question of density construction, which can be done not uniquely, still remains. The easiest and mostly natural way is to search the density among the piecewise constant functions which, together with the previous remark on symmetry, leads to representation of particles by parallelepipeds (prisms or cylinders) the centers of which are the nodes of quadrature formula obtained as the result of moving of initial nodes. Some part of the parallelepiped side surface has to be "contiguous" to a nearest neighbor. Hence we understand a "particle" as a cylinder (the base of which can be an arbitrary figure) having the volume $1/N$ (or c_i), the center in $z(t, \xi_i^{\mu_0})$ and the height equal to the measure density in its center.

It is not clear from above how large has to be the particle base and hence its height, i.e. the measure density. The "contiguity" can be determined if the dimensions of the particle bases are already defined but it is not so at any new time moment. Different iterative procedures for the problem solving can be proposed. We propose to use an explicit scheme: having the information on the particles dimensions at the previous time moment, we shift them and after that change their dimensions demanding the "non – overlaping" of the particles for that the set of the changed particles is an approximation of a measure which is the approximate solution of the original task.

We shall change the particles dimensions according to the following rule of *the couple interaction*:

if two particles "overlaps" then the dimension of the low particle base is decreased that leads to increase of its height and decrease of the density gradient as usually occurs in physical phenomena:

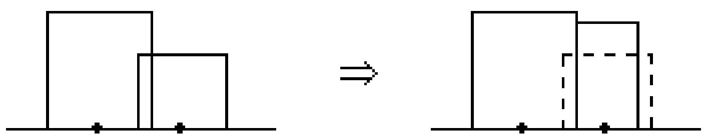

Figure 1: Overlaping of two particles and its processing.

if a particle becomes isolated from its neighbours then the nearest one is determined and the base dimensions of one of this couple participants (the one which is higher) is increased and its height becomes less:

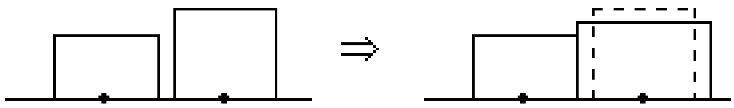

Figure 2: Isolation of two particles and its processing.

3 QUASI – LINEAR EQUATION

Assume that the measure μ_t has a density $f_t(z)$. Consider the problem (2) with the right hand side

$$F(t, z(t)) = \frac{1}{2} f_t(z(t)).$$

Then the measure μ_t built according the formula (4)

$$\langle \psi, \mu_t \rangle = \int \psi(z(t, \xi)) \mu_0(d\xi)$$

satisfies the system of equations

$$D_t \langle \psi, \mu_t \rangle = \langle D\psi[\frac{1}{2} f_t], \mu_t \rangle, \quad \mu_{+0} = \mu_0, \tag{7}$$
$$\langle \psi, \mu_t \rangle = \int \psi(z) f_t(z) dz,$$

which is nearly a generalized quasi – linear transport equation with the restriction that its solution (the measure μ_t) possesses a density $f_t(z)$. This restriction expresses the hart of the ill — posed problem regularization with the help of particle method. Let's clarify the last statement.

It is well known (f.e., [9]) that Cauchy problem is ill – posed because of non – uniqueness of its solution. The problem of constructing a well — posed class of solutions led to introduction of additional so called "entropy" condition [7, 8, 9] and building a "method of disappearing viscosity" [8] and "method of smoothing" [9] for proving the theorems of existence and uniqueness.

The algorithm of particle method for the system of equations (7) coincides in general with the one for linear transport equation. The difference is in that the right hand side of (2) is not a given function but depends on the solution.

The way of recovering the measure density by the configuration and distribution of the particle weights is described at the end of the last paragraph. In case of quasi – linear equation it has to be supplemented with the check of condition that the height of the particle, which has rose as the result of interaction with another particle, has not to be greater than the height of the latter. This condition appears from the comparing of the discrete algorithm behaviour to exact solutions of test problems, their physical meaning [13, 14, 15, 18] and is similar to the introduction of axioms of "entropy" condition type that leads to the maximum principle and variation boundedness of quasi – linear equation solutions [7, 8, 9]. In case of 1D quasi – linear equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

"Oleinik's condition" looks simple and clear:

$$\min(u^-, u^+) \leq u \leq \max(u^-, u^+),$$

where $u^{\pm}$ are the solution values from left and right of the discontinuity which can be taken in particle method as the heights of neighbour particles. The violation of this condition demands the change of the reconstructing particle height till the level of the particle which induced the describing change. To conserve its volume it is necessary to increase the square of its base that, combining with "non – overlaping" requirement, leads to the shift in the direction of the vector connecting the centers of these two particles:

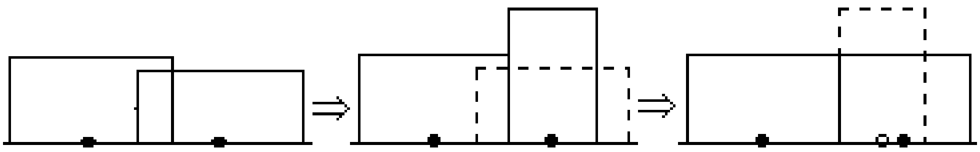

Figure 3: Application of "entropy" condition.

Our algorithm can be regarded as a concrete definition of "Oleinik's condition", namely, a requirement for a magnitude to belong to an interval turns into specific value of the magnitude.

Finally we have the algorithm for solving a problem discretized with the help of representation of weak generalized solution (the measure μ_t) by a set of N particles, the essential part of which is the method of the measure density construction. The recovering of the density measure by the discrete information can be considered as a regularization of the original task. The validity of the described regularization method is a consequence of the above considerations and is supported by a number of computational experiments.

Recently a result on convergence of a particle method (called Finite Volume Particle Method) [16], which is similar to the Method of Large Particles [17], for a scalar quasi – linear equation was obtained. Here the regularization is reached by the account of finite difference flux through the particle boundaries with the help of introduction of additional terms localized at particle bases overlaps. These terms play the role of smoothing. The authors also remark the effect of particles overlaping but overcome the problem by the way different to our one and quite sophisticated from algorithm view point. How it will work for concrete computational situations, how useful is this regularization will be seen, probably, from the further study.

4 SYSTEMS OF EQUATIONS

For gas dynamics system of equations enabling discontinuous solutions, we are not aware (as well as [9]) of the mathematical results on construction a well – posed class of solutions similar to the way how it was done for one equation with the help of "entropy" condition.

There are some papers on particle method substantiation in cases of smooth solutions for Vlasov system of equations [10, 3] and for Euhler equations system [11]. The last paper can be considered as a mathematical base for SPH method.

Our approach differs from the other particle methods by the two main features: first, we approximate the mass, impulse and energy densities of a gas by three different particle sets which evolves according to appropriate conservation laws by the algorithm described above for 1D equation and, second, we apply certain procedures to maintain well–posed solutions of the problem which enables discontinuous behaviour.

To reduce writing we illustrate our ideas at an isothermic non viscous (isentropic) 1D problem (simplest but not trivial):

$$\begin{aligned} \frac{\partial}{\partial t}\rho &= -\nabla \cdot (\rho v), \\ \frac{\partial}{\partial t}(\rho v) &= -\nabla \cdot (v \rho v) + P, \\ P &= \sigma(\rho), \end{aligned}$$

where P denotes some imaginary force which leads to the impulse density change because of its redistribution among the neighbour space cells as the result of heat moving of the molecules; this force is proportional to a gradient of some magnitude p called pressure and determined from the thermodynamics as a function σ depending in our case only on ρ. The consideration of additional viscosity, boundary conditions, outer forces and even incompressible fluid has no principal barriers and is also done [15, 18].

We study a generalized problem statement restricted by the assumption of existence of densities for unknown measures:

$$\begin{aligned} D_t\langle \psi, \mu_t^{\rho}\rangle &= \langle D\psi[v], \mu_t^{\rho}\rangle, \quad \mu_{+0}^{\rho} = \mu_0^{\rho}, \\ D_t\langle \psi, \mu_t^{\rho v}\rangle &= \langle D\psi[v], \mu_t^{\rho v}\rangle + \langle \psi, P\rangle, \quad \mu_{+0}^{\rho v} = \mu_0^{\rho v}, \\ \langle \psi, \mu_t^{\rho}\rangle = \int \psi(z)\rho(z)dz, \quad \langle \psi, \mu_t^{\rho v}\rangle &= \int \psi(z)(\rho v)(z)dz, \quad v = \rho v/\rho. \end{aligned} \tag{8}$$

The evolution of the measures μ_t^ρ and $\mu_t^{\rho v}$ proceeds according to space evolution of generating processes $z^\rho(t)$ and $z^{\rho v}(t)$:

$$D_t z(t) = v(t, z(t)), \quad z\,|_{t=0} = z_0.$$

The presence of the force P in the right hand side of equation for impulse leads to the latter redistribution among the neighbour particles which approximate the measure $\mu_t^{\rho v}$ in accordance to the specific form of the force P, f.e. in the isentropic shock wave problem it is defined only by pressure gradient whereas in the shallow water models the gravity force should be added. Such a redistribution influences only the particle volumes and not their positions.

So the algorithm of the particle method consists in their shift, change of their forms and volumes. While doing these operations we have to follow the rules similar to described above but with some new features taking into account the higher complicity of the problem. Let's discuss it more carefully.

At an initial time moment we discretize the measures μ_0^ρ and $\mu_0^{\rho v}$ by the two particle sets N^ρ and $N^{\rho v}$ (let them both be equal to N for simplicity; the same is usually done in computations). So we get the two sets of initial positions, volumes and forms of the particles.

Further on, introducing the time grid, we shift the particles according the explicit Euhler scheme (the implicit schemes as well as the schemes of higher accuracy can also be applied, but our experiments showed that this stage is not the key one in our approach).

The second, important, stage, in which the peculiarity of our method is just consists, is the reconstruction of the particle forms and the volume change of the particles representing the impulse.

As the result of the shift the particles can overlap or fly away from each other. This effect violate the approximation of the measures by the discrete particle sets. To avoid it we change the particle forms in a way that, first, they will stay symmetric corresponding to their centers and, second, they have to be "contiguous" to their nearest neighbors as we already described above concerning the transport equation. The particles – candidates to form change are determined by the same as above selection rules which gain a clear physical meaning in the case of concrete model: at overlaping of the particles, modelling the masses in the continuity equation, the particle with lower density will be changed (will be narrowed and grown). The "entropy" condition, which seems quite unnatural for quasi – linear transport equation, in this case becomes absolutely transparent: after its growth the height of the particle (the mass density) cannot exceed the height (the density) of the one which was the reason of this growth. The same considerations are valid for the particles modelling the impulse with the only difference that instead of the mass density we have to take into account the physical behaviour of the velocity module.

It is also important how to change the particle volumes. This procedure is determined by the thermodynamic properties, i.e. by the equation of state and the mechanism of impulse exchange between the neighbour gas cells. Additionally a rule according to which the impulse transfers only between the contiguous particles is introduced [13, 14]. Without it the solution will be much smoother.

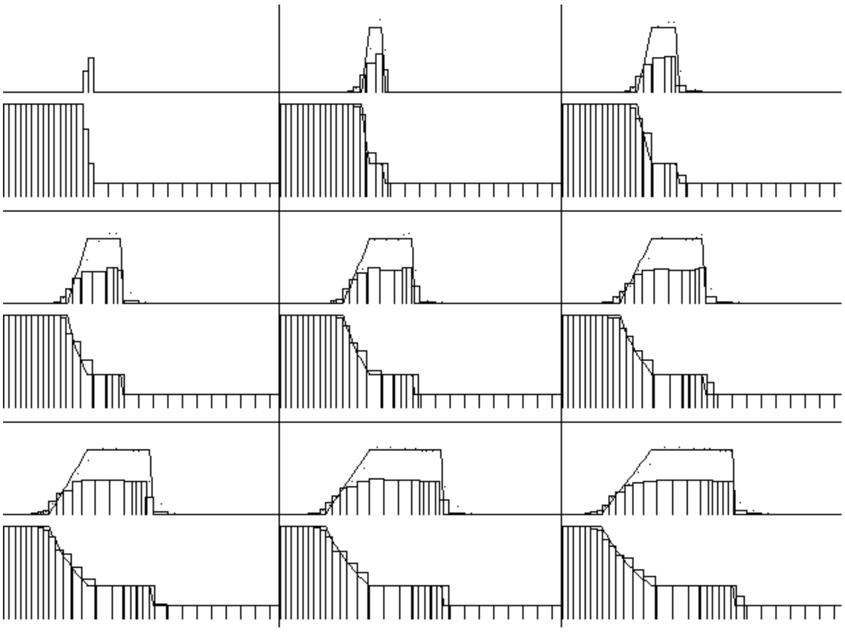

Figure 4: Riemann problem, $N = 30$, $\Delta t = 0,001$

We illustrate our method at the problem with discontinuous initial data (Riemann problem). The index "2" denotes the values of ρ and ρv left from the discontinuity, and the index "1" is prescribed for the right ones. The dimensionless sound speed $c = 1$. The density ρ_1 is a free parameter. Then the exact solution can be written as

$$\rho_2 = \rho_1 \cdot e \cdot \left(\left(3 + \sqrt{5}\right)/2\right),$$

in the rarefaction wave the velocity changes from 0 to the sound speed and the density varies from ρ_2 to ρ_2/e according to:

$v\,(x,t) = x/t + c \quad \rho\,(x,t) = \rho_2 \cdot exp\,(-\,(x/t + c)).$

The velocity of the shock wave front is

$$D = v_1 + \sqrt{(\rho_2/e)/\rho_1}.$$

To calculate this problem we applied the splitting method: first, we shifted and reconstructed the particles – masses, second, we did the same to the particles – impulses and, third, we changed the latter volumes. There are some computational details, f.e. how to deal with overlaping mass particles having equal heights (densities), how to proper calculate the pressure gradient etc. Unfortunately there is no place here for them.

The computation results are shown at Fig.4 and 5 for nine different time moments from left to right. At the upper parts of the pictures the dots represent the velocity values of the particles, the small rectangles means the particles – impulses; at the lower parts of the pictures the evolution of the mass density by rectangles – masses is reproduced. The continuous lines are the exact velocity and mass density. The number of particles $N = 30$.

At Fig.4 the solution with the time step $\Delta t = 0,001$ after each 20 steps is presented. At Fig.5 $\Delta t = 0,01$ which nearly corresponds to Courant number if to regard a width of a particle as a difference space cell. Let us underline that the shock wave front is spread at only one particle. This fact does not depend on particle number. We took such a small particle number to emphasize the method possibilities.

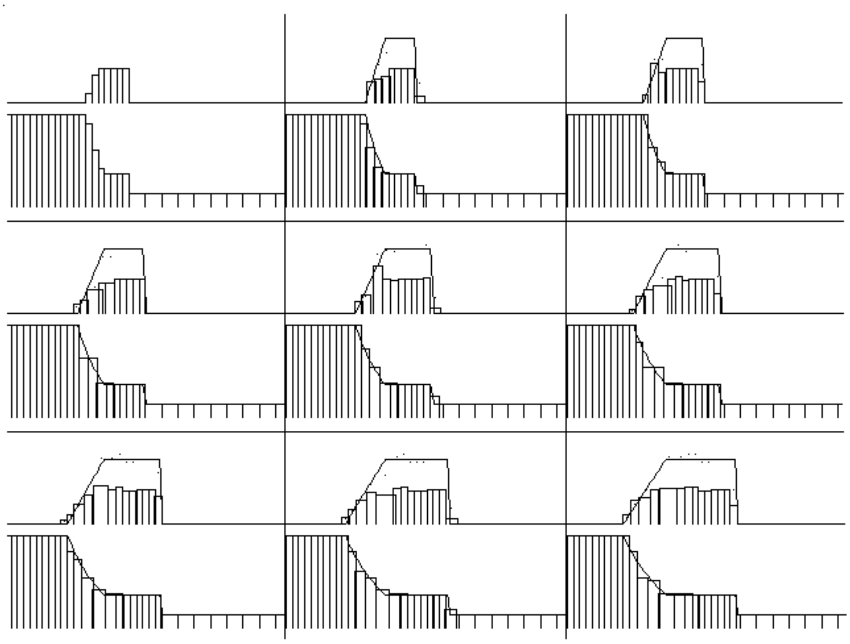

Figure 5: Riemann problem, $N = 30$, $\Delta t = 0,01$

We also tested our method at shallow water models [15]. The results for the problem of sudden flood wave propagation are presented at Fig.6 and 7 for the number of particles $N = 22$ and $N = 200$. The exact solutions for water velocity and height are drawn by the continuous lines.

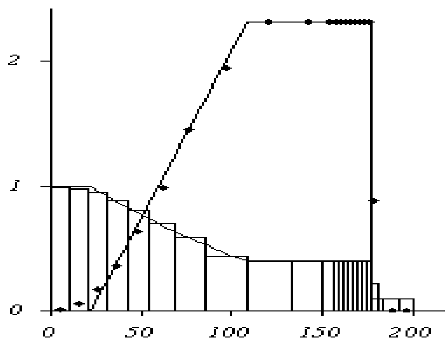

Figure 6: Sudden flood wave propagation, $N = 22$

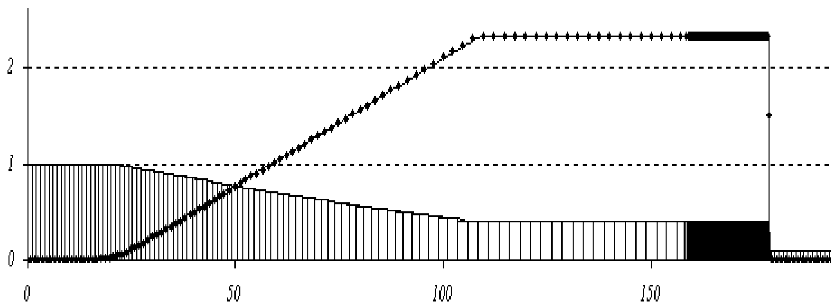

Figure 7: Sudden flood wave propagation, $N = 200$

5 2D INCOMPRESSIBLE FLUID

The described principles works even for incompressible viscous fluid [18] which is not an easy task for the particle method.

As a test let us consider Blasius problem of a flow past a flat plate:

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{9}$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{10}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial u\rho}{\partial x} + \frac{\partial v\rho}{\partial y} = 0 \tag{11}$$

with boundary conditions $(u = v = 0)$ at the plate and $u = 1, v = 0$ in coming flow; Re is Reinolds number.

The classical asymptotic solution gives the width of boundary layer as

$$\delta_b(x) = 5,16\sqrt{\frac{1}{Re}x/U_\infty},$$

where the flow velocity at infinity $U_\infty = 1$. Inside the boundary layer x-component of the velocity (u) varies from 0 to U_∞ according the parabolic law and y-component (v) is determined by the given x-one in the whole region from the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

We intentionally conserved the constant density ρ in (11) because it is important for our approach.

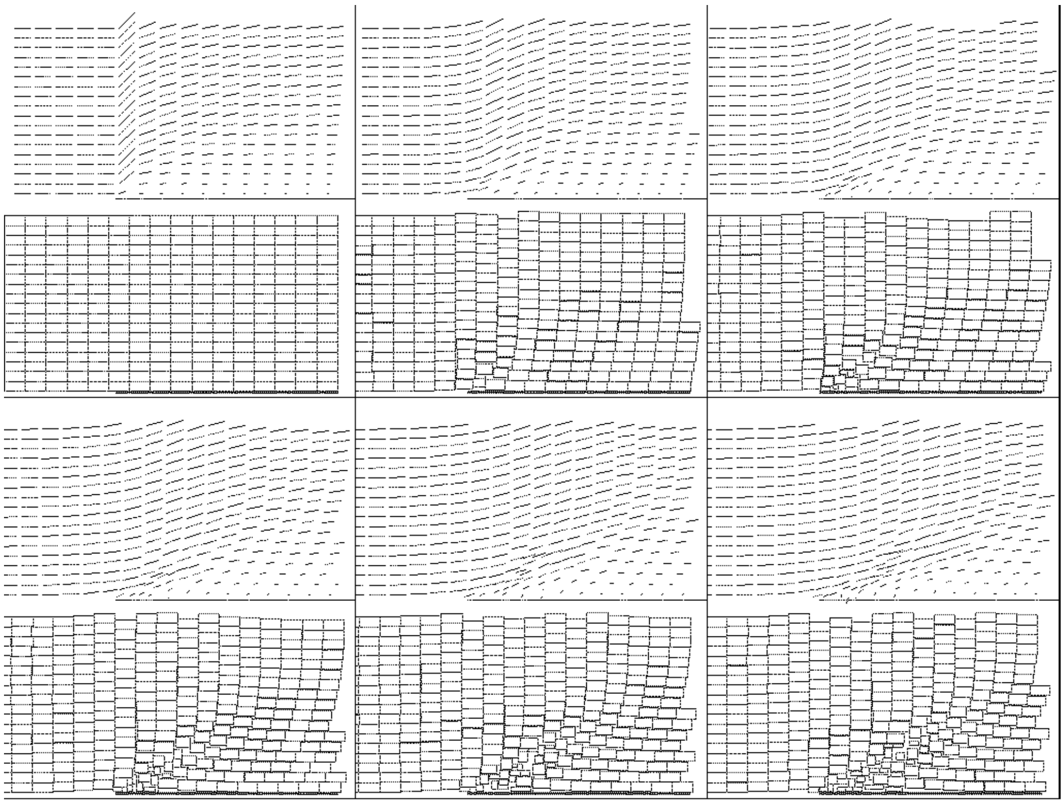

Figure 8: Blasius problem.

The main difference of Navier – Stokes system of equations for incompressible fluid from the one for compressible gas is the absence of the equation of state. It involves some unnatural features in physical interpretation of the problem statement: the continuity equation turns into a restriction at velocity components and the equations for velocity components turns into the equations for determination of a value called pressure.

To implement the particle method we have to calculate the evolution of particles – masses, taking into account that the squares (not the forms) of their basis cannot change because of incompressibility, and particles – impulses, changing their volumes according to the terms connected to the pressure. We have no the equation of state for that but we know that there are no discontinuous solutions in incompressible fluid. It gives the possibility to redistribute the impulse between the neighbor particles – impulses according to the magnitude of the overlap of the corresponding particles – masses. The impulse have to flow from the particle with higher impulse module to the particle with the lower one in the direction connecting their centers like in the problem of two bodies elastic interaction. The viscosity changes the tangential components of the impulses. To fulfill all these operations we use the splitting method though an explicit scheme may be sufficient. From the other hand, may be, it is better to use implicit scheme because of very high speed of sound in the incompressible fluid. To solve the nonlinear transport equations in 2D the splitting is not necessary.

Because of the high singularity of Blasius problem in the region of the plate nose we also had to include the procedure of particles "birth and death".

The particles have to be sorted for the reduction of operations number. A specific algorithm for the boundary conditions is implemented.

We announced the main ideas of our method. The computational results are presented at Fig.8 and 9. Fig.8 represents the x-component of the velocity, u, of the flow at six time moments separated by 15 steps equal to 0,003, $1/Re = 0.05$, the initial sizes of 288 particles are approximately 0.063×0.056. At the upper parts of the drawings 2D vectors (without arrows because their directions are clear) of u are shown; at the lower parts we put the distributions of our rectangular particles – impulses (view from above). At Fig.9 the dark ellipses show the difference between numerical and benchmark solution of our test problem: it is obvious that the difference has to be large at the nose of the plate.

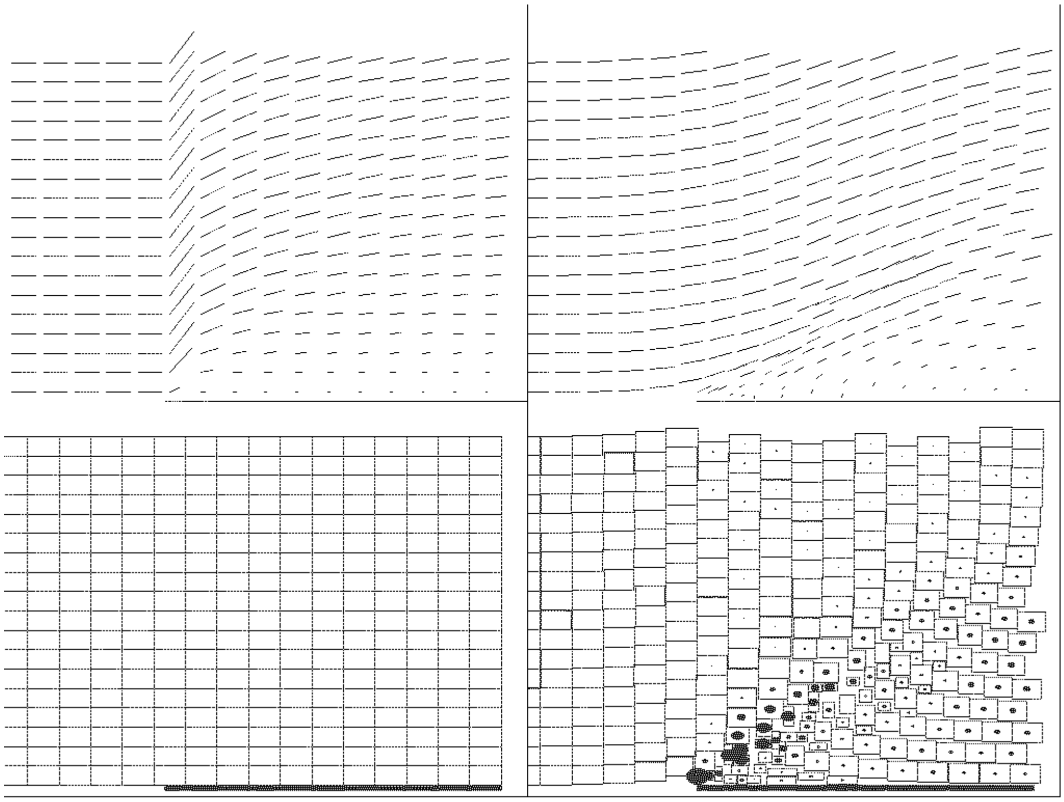

Figure 9: Blasius problem, comparison with benchmark solution.

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