Impedance Calculation of Multiconductor Transmission Lines Including Superconductors

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1 Introduction

1.1 Problem definition

The transmission line model for stretched, homogeneous system of conductors is considered. In this model the total currents in the conductors and conductor voltages are connected by one of two of frequency domain transmission line (or telegrapher's) equations:

$$\frac{dV}{dz} = -\mathbf{Z}(\omega)\mathbf{I} \tag{1}$$

where

$$\mathbf{Z}(\omega) = \mathbf{R}(\omega) + j\omega\mathbf{L}(\omega). \tag{2}$$

$\mathbf{Z}$, $\mathbf{R}(\omega)$ and $\mathbf{L}(\omega)$ are frequency dependent series per-unit length impedance, resistance and inductance matrices. z is longitudinal direction along transmission line

Our purpose is to develop computer program for effective impedance matrix calculation under very general assumptions concerning shape of cross-section of conductors and conductor material properties. In particular we want to consider superconductors - normal conductors transmission lines.

1.2 History notes

The problem of inductance calculation of long parallel wires is well-known. The detailed discussion of the physical statement of the problem for normal conductors can be found in [1]. Different applications are discussed in [2, 3, 4, 5]. Cable power transmission problems are considered, for example, in [6, 7] and in many other works.

For superconductors inductance and impedance extraction problems were discussed in [8, 9, 10].

Analytical methods for impedance estimation are very limited. Only simple problems can be solved [11, 12].

Significant efforts were applied last decades for the numerical transmission line impedance calculation. Most physically evident and simple is volume integral equation method ([11, 13] -normal conductors, [8, 9, 10] superconductors). In this approach the integral equation in form of volume potential for current density is written over the cross-section of the conductors. Unfortunately this method has inherent disadvantages. For small skin depth volume integral equations are ill-conditioned [13, 14] and calculations can be numerically unstable. Besides that this approach leads to systems of linear equations with very large dimension even for simple problems. All known realizations of the volume integral equation method accept only rectangular or combined from rectangles cross-sections of conductors.

In many papers [2, 3, 4, 5] finite element method is used. Finite element and finite differences methods are the only possible techniques for problems with variable coefficients. But these methods has certain specific problems and limitations. Finite element equations are not local because they contain integral terms with solution integration over cross-section of conductors [2, 3, 4, 5]. This fact leads to sparse matrices with large dense blocks indiscrete finite elements equations. It leads to more complicated and less effective numerical technique. Besides that finite element or finite differences methods has serious problems for open enviroment problems where solution need to be found in unbounded domain. For example good finite differences numerical technique [8] was found not suitable for practical problems in [15]. Because of this obstacle finite element method can not be effectively applied for shielding analysis problems.

Most promising technique is Boundary Element Method (BEM). The only limitation of this technique is applicability for constant coefficients problems. In our case it means that conductivity need to be constant or piece-wise constant. It is quite suitable assumption. BEM was applied in different forms in many works. Technique close to direct BEM based on surface electric and magnetic currents on conductor boundary was developeted in [1]. An attempt for BEM application was done in [17] where wrong form of 3rd Green identity was used. Complete realization of BEM in classic [16] form was done in [18], but impedances were not considered there. In [1, 17, 18] superconductors were not considered.

1.3 What we need

We would like to develop new version of BEM program with next features:

1. Arbitrary form of cross-sections of the transmission lines;
2. Conductors made from different metals;
3. Superconductors if necessary combined with normal conductors;
4. Shielding problems;
5. Cables;
6. Modern, fast and reliable numerical technique;
7. Brief and effective input data;
8. Portable effective program.

In this paper we outline theory and technique used for development of our program TLZ which meets the above requirements.

2 Theory

2.1 Normal conductors

Consider Maxwell equations for conductors (neglecting displacement current):

$$\nabla \times \mathbf{B} = \mu\mu_0 \mathbf{J}, \tag{3}$$

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}, \tag{4}$$

where $\mathbf{B}$, $\mathbf{E}$ - magnetic and electric fields, $\mathbf{J}$ - current, ω - angular frequency, μ is relative magnetic permittivity. Current obeys Ohm's law

$$\mathbf{J} = \sigma \mathbf{E}. \tag{5}$$

Consider vector potential of magnetic field $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad \nabla \cdot \mathbf{A} = 0. \tag{6}$$

Then

$$\mathbf{E} = -j\omega \mathbf{A} + \nabla V. \tag{7}$$

Excluding current we obtain well-known equation for vector potential

$$\Delta \mathbf{A} + k^2 \mathbf{A} = k^2 \frac{\nabla V}{j\omega}, \tag{8}$$

$$k^2 = -j\omega\mu\mu_0\sigma. \tag{9}$$

In this equation ∇V need to be considered as excitation external voltage drop. For further convenience consider function φ:

$$\varphi = \frac{V}{j\omega}. \tag{10}$$

Outside the conductors (in the embedding media) the equation for vector potential is Laplace equation

$$\Delta \mathbf{A} = 0. \tag{11}$$

The equations need to bee fulfilled by certain junction conditions on the boundary of conductors and embedding media, the limiting condition for vector potential on the infinity and certain excitation conditions.

2.2 Superconductors

Consider two-fluid model [19] of electrical current in the superconductors:

$$\mathbf{J} = \mathbf{J}_s + \mathbf{J}_n, \tag{12}$$

where $\mathbf{J}_n$ is the normal current and $\mathbf{J_s}$ is the superconductivity current. The normal current obeys Ohm's law

$$\mathbf{J}_n = \sigma_n \mathbf{E}, \tag{13}$$

where $\mathbf{E}$ is the electric field in the superconductor and σ_n is the conductivity. For superconductivity current London equations are used:

$$\mathbf{J}_s = \frac{1}{j\omega\mu_0\lambda^2}\mathbf{E}, \tag{14}$$

$$\mu_0\lambda^2\nabla \times \mathbf{J}_s + \mathbf{B} = 0, \tag{15}$$

where $\mathbf{B}$ is the magnetic field and λ is London penetration depth. The other equations are Maxwell equation for the magnetic field neglecting displacement current:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_s + \mathbf{J}_n), \tag{16}$$

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}. \tag{17}$$

For vector potential $\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$ with the gauge $\nabla \cdot \mathbf{A} = 0$ the London equation is:

$$\mu_0\lambda^2\mathbf{J}_s + \mathbf{A} = \nabla\varphi, \tag{18}$$

where φ is phase:

$$\Delta\varphi = 0, \qquad \nabla V = j\omega\nabla\varphi. \tag{19}$$

The relation between phase and voltages is

$$\nabla V = j\omega\nabla\varphi. \tag{20}$$

Excluding current density we obtain

$$\Delta\mathbf{A} + k^2\mathbf{A} = k^2\nabla\varphi, \tag{21}$$

$$k^2 = -\frac{1}{\lambda^2} - j\omega\mu_0\sigma_n. \tag{22}$$

We assume the superconductors are buried in homogeneous media with magnetic permittivity μ_0. The equation for vector potential is Laplace equation:

$$\Delta\mathbf{A} = 0. \tag{23}$$

These equations need to bee fulfilled by certain junction conditions on the boundary of superconductors, the limiting condition on the infinity and excitation conditions. We discuss these conditions in details for the 2D problem.

2.3 Transmission Line Problem

Now we need to consider 2D distributions of current (normal or superconductivity) $\mathbf{J} = \{0, 0, J(x, y)\}$ and vector potential $\mathbf{A} = \{0, 0, A(x, y)\}$ in stretched homogeneous transmission line. Then vector equation for vector potential $\mathbf{A}$ is reduced to single scalar equation with constant right part because for 2D problem

$$\varphi = z\psi(x, y), \quad \nabla\psi(x, y) = 0. \tag{24}$$

Then

$$(\nabla\varphi(x, y))_z = \Phi \equiv const. \tag{25}$$

These constants are different for different conductors.

Let us write accurately equations for conductors and embedding media. Let the number of conductors be $N+1$. The conductor with number 0 is presenting the groundplane and is carrying return current. In some cases groundplane conductor can be "empty". Each conductor can contain one or some wires. Each wire is characterized by wave constant k in the equation for vector potential. Let S_m be the cross-section of m-th conductor, Γ_m - its boundary, $r = (x, y)$. Then full current in m-th conductor is

$$I_m = \int \int_{S_m} J(r) dx dy. \tag{26}$$

Let A_m and Φ_m be the values for m -th conductor, then we have the next problem:

$$\Delta A_m + k^2 A_m = k^2 \Phi_m, \quad r \in S_m, \quad m = 0, \ldots, N; \tag{27}$$

$$\Delta A = 0, \quad r \in R^2 \setminus \bigcup_{m=0}^{N} S_m, \tag{28}$$

$$A_m(r) = A(r), \quad \frac{\partial A_m}{\partial n} = \mu_m \frac{\partial A}{\partial n}, \quad r \in \Gamma_m; \tag{29}$$

$$\int_{\Gamma_m} \frac{\partial A}{\partial n} ds = \mu_0 I_m, \quad m = 0, \ldots, N. \tag{30}$$

Here n is the internal normal to Γ_m relatively to the domain S_m. The junction conditions follows from continuity of vector potential and it's normal derivative on Γ_m. In these equations Φ_m are unknowns and values I_m are given, $m = 0, \ldots, N$.

What concerns conditions for $A(r)$ at infinity, we will search the solution in the form

$$A(r) = \ln \frac{1}{r} \sum_{m=0}^{N} I_m + A_\infty + O(\frac{1}{r}), \tag{31}$$

where

$$A_\infty = 0. \tag{32}$$

For problems with groundplane carrying all return current the first term with logarithm growth is zero.

2.4 Impedance matrix

As our problem is linear then exist $N \times N$ complex matrix M that

$$\Phi_i - \Phi_0 = \sum_{j=1}^{N} M_{ij} I_j. \tag{33}$$

Matrix M is the matrix of complex inductances. Impedance matrix $Z(\omega)$ is

$$Z(\omega) = j\omega M(\omega). \tag{34}$$

3 Boundary integral equations and numerical technique

3.1 Volume integral equations

There are several possibilities to reduce the system of differential equations to integral equations. The most evident approach is to reduce the problem to system of $N+1$ volume integral equations for functions $J_m(r)$. Let us present $A_m(r)$ as logarithmic volume potential with density $J_m(r)$. Then we obtain:

$$\frac{1}{k^2}J_m(r_0) - \frac{1}{2\pi}\sum_{l=0}^{N}\int\int_{S_l} \ln(|r-r_0|)J_l(r)dxdy - \Phi_m = 0, \tag{35}$$

$$\int\int_{S_m} J_m(r)dxdy = I_m, \qquad m = 0, \dots, N. \tag{36}$$

It can be easily shown that the variational method used in [9] and net approach and [10, 11] coincides with the Galerkin algorithm for these integral equations [13]. The numerical solution of volume integral equations is unstable for high frequencies because the out-integral term with k^2 becomes small. The instability of numerical solution was reported in [10, 17] and can be explained by ill-conditionality of volume integral equations.

Another disadvantage of volume integral equations approach is large number of unknowns in this method. As result in many cases calculations can be slow. The reason of large number of unknowns is the necessity to divide in mesh cells the total area of conductors cross-sections.

3.2 Boundary Integral Equations (BIE)

The reduction of our differential equations problem to boundary integral equations is essentially based on the third Green's identity technique [20, 16, 21, 13].

Consider linear second order differential operator L in bounded domain Ω in R^2 with boundary $\partial\Omega$. In our case L can be or Laplace's or Helmholtz operator. Let $n(r)$ - internal normal to $\partial\Omega$ in point $r \in \partial\Omega$ and

$$\partial = \frac{\partial}{\partial n(r)}. \tag{37}$$

Let $q(r, r_0)$ is the fundamental solution of differential equation

$$Lq(r, r_0) = 2\pi\delta(r - r_0) \tag{38}$$

in R^2, $\delta(r)$ - delta-function. Then the third Green's identity has the form:

$$\omega(r_0)u(r_0) - \int_{\partial\Omega}(u(r)\partial q(r,r_0) - q(r,r_0)\partial u(r))ds_r + \int\int_{\Omega} Lu(r)q(r,r_0)dxdy = 0, \tag{39}$$

where $\omega(r_0)$ is the visibility angle of curve $\partial\Omega$ from the point r_0, $\omega(r_0) = 2\pi$ if r_0 is located inside Ω and $\omega(r_0) = \pi$ if r_0 is non-corner point of $\partial\Omega$.

For Lapalce's equation

$$q(r, r_0) = G(r, r_0) = -\ln|r - r_0|. \tag{40}$$

For the unbounded domain under the condition $A_\infty = 0$ third Green identity can be rewritten in the form ($u(r) \equiv A(r)$):

$$\omega(r_0)A(r_0) + \int_{\partial\Omega} (A(r)\partial G(r, r_0) - G(r, r_0)\partial A(r))ds_r = 0. \quad (41)$$

For Helmholtz equation $L = \Delta + k^2$ the suitable form of fundamental solution $q(r, r_0)$ is

$$q(r, r_0) = M_m(r, r_0) = -\frac{j\pi}{2} H_0^{(2)}(k|r - r_0|), \quad (42)$$

where $H_0^{(2)}(r)$ is the Hankel function and condition $Im(k) < 0$ is assumed. For superconductors and $\omega = 0$ (stationary case) kernel is reduced to

$$M_k(r, r_0) = K_0(\frac{|r - r_0|}{\lambda}), \quad (43)$$

where $K_0(r)$ is the modified Bessel's function.

Writing 3rd Green identity for $u(r) = A_k(r) + \Phi_k$ we obtain:

$$\omega(r_0)(A_m(r_0) + \Phi_m) - \int_{\Gamma_m} ((A_m(r) + \Phi_m)\partial M_m(r, r_0) - \partial A_m(r)M_m(r, r_0))ds_r = 0. \quad (44)$$

Taking into account the continuity of $A(r)$ and $\partial A(r)$ on the boundary of every conductor we obtain the following system of boundary integral equations for $A_m(r)$ and $\partial A_m(r)$:

$$\pi A_m(r_0) + \sum_{l=0}^{N} \int_{\Gamma_l} (A_l(r)\partial G(r, r_0) - \partial A_l(r)G(r, r_0))ds_r = 0, \quad (45)$$

$$\pi A_m(r_0) - \int_{\Gamma_m} (A_m(r)\partial M_m(r, r_0) - \partial A_m(r)M_m(r, r_0))ds_r + (\pi - \int_{\Gamma_m} \partial M_m(r, r_0)ds_r)\Phi_m = 0, \quad (46)$$

$$\int_{\Gamma_m} \partial A_m(r)ds_r = \mu_0 I_m, \quad r_0 \in \Gamma_m, \quad m = 0, \ldots, N, \quad (47)$$

where r_0 is non-corner point of Γ_m.

The derived system of boundary integral equations provides more effective approach to the numerical solution of the problem.

3.3 Numerical technique

For numerical solution of BIE we use method of approximation of integral operators in main based on general scheme of direct boundary element method outlined in [21]. More details about numerical method can be found in [14].

Let h is the mean step of mesh on curves Γ_m. We define approximate values of $A_m(r)$ and $\partial A_m(r)$ in the nodes of nearly uniform mesh on Γ_m, $m = 0, \ldots, N$. The total number of unknowns in our case is $O(1/h)$. For bulk conductors it is in order of magnitude less then in volume integral equations methods.

Our method results in linear system of equations with block dense matrix. Some blocks of the matrix contain zeros, because equation for each conductor involve the values for this conductor only. The structure of this sparsity is well-known [16]. Really in our program we do not exploit the block sparsity of the matrix and use ordinary Gauss elimination in the form of LU decomposition. Even in this case the whole algorithm appears to be fast enough.

As follows from results [14, 22], the expected precision of our method need to be $O(1/h^2)$. The second order of the convergence was verified by set of calculations for variety of examples by densing the mesh.

4 Numerical examples

4.1 Two-wire coaxial line

First example concerns the problem with analytical solution. It is single core shielded cable. The cross-section of inner wire is circle and the cross-section of shield is coil with coinciding centers. Then the solution can be easily expressed using Bessel functions.

For different values of size of the cross-sections for wide range of frequencies the excellent agreement of analytical and numerical solution was obtained. Program exactly reproduce the symmetry of the solution. The typical results are presented in the Table 1.

	Numerical		Analytical	
f Hz	$R \cdot 10^6 \Omega/m$	$L \cdot 10^6 H/m$	$R \cdot 10^6 \Omega/m$	$L \cdot 10^6 H/m$
10^2	6821.06	0.221	6821.04	0.221
10^3	6832.68	0.221	6832.65	0.221
10^4	7861.11	0.216	7861.06	0.216
10^5	21234.3	0.17	21234.2	0.17

Table 1: Coaxial shielded cable, wire radius $1mm$, shield radiuses $2mm$ and $3mm$, conductivity $56MS/m$.

The time of calculations is negligible because the results were obtained for small number of mesh points.

4.2 Rectangular conductors

Consider two rectangular conductors, $[0, 2] \times [-0.6, -0.4]$ and $[0, 2] \times [0.4, 0.6]$ mm with conductivity $56MS/m$. It is well-known example from [1]. The results of our calculations are presented in the Table 2.

Small difference in the results can be explained by the approximate value of the excitation in [1], p. 964. Our results were obtained on the mesh with 8 steps per $1mm$ and it took 5 seconds on Celeron-366 IBM PC.

Another example from [1] deals with two bulk (or "thick") rectangular conductors $[0, 2] \times [-3.0, -1.0]$ and $[0, 2] \times [1.0, 3.0]$ mm. In this example skin depth can be much less then linear size of conductors. The results for conductivity $56MS/m$ are presented in Table 3. The results for two independent approaches and programs coincide with good accuracy.

	Our results		Djordjevic et. al.	
f Hz	$R \cdot 10^6 \Omega/m$	$L \cdot 10^6 H/m$	$R \cdot 10^6 \Omega/m$	$L \cdot 10^6 H/m$
10^3	89140	0.357	87600	0.357
10^6	181665	0.335	175300	0.334
10^9	$5.688 \cdot 10^6$	0.311	$5.79 \cdot 10^6$	0.310

Table 2: Two conductors with long thin rectangle cross-sections [1](1), conductivity $56MS/m$, comparison.

	Our results		Djordjevic et. al.	
f Hz	$R \cdot 10^6 \Omega/m$	$L \cdot 10^6 H/m$	$R \cdot 10^6 \Omega/m$	$L \cdot 10^6 H/m$
10^2	8915.99	0.5897	8780	0.586
10^3	8937.65	0.596	8780	0.588
10^4	11219.6	0.577	11020	0.572
10^5	31593.8	0.497	30800	0.496
10^6	88274.2	0.466	86000	0.466
10^7	273847	0.457	273200	0.456
10^8	898542	0.454	874600	0.454

Table 3: Two conductors with long thin rectangle cross-sections [1](2), conductivity $56MS/m$, comparison.

4.3 Superconductors

Our program accurately reproduce all static results from [8, 9, 14]. Example 1 test which has analytical solution for two-fluid model also was solved with high accuracy.

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